INFORMATION AND CONTROL 45, 34–36 (1980)

A Note on the Upper Bound for the Difference between Two Entropies

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In a previous paper, the author proved that the difference between the Shannon entropies of the original and the q-deleted point process is bounded above by a function of q. This note strengthens the result by showing that the bound cannot be improved and is in fact the least upper bound for the difference.

1. INTRODUCTION

Let $\{p_n\}$ $(n = 0, 1,...)$ be any distribution over the non-negative integers and let us define the distribution $\{p_{q,n}\}$, $0 < q < 1$, by

$$p_{q,n} = \sum_{m=n}^{\infty} \binom{m}{n} q^n (1-q)^{m-n} p_m . \tag{1}$$

One interpretation of the distribution defined by (1) can be given in the context of independent deletions of a point process. Consider a point process occurring in an arbitrary set X and let p_n $(n = 0, 1,...)$ be the probability of getting exactly n points in X. Then $p_{q,n}$ is the probability of n points remaining in X after each point of the original process is randomly deleted with probability $1 - q$ and left untouched with probability q.

Zeephongsekul (1978) has considered some properties of the Shannon entropy

$$I_q(P) = -\sum_{n=0}^{\infty} p_{q,n} \ln p_{q,n} . \tag{2}$$

In particular (Zeephongsekul, 1978, Theorem 2) we proved that for any initial distribution $\{p_n\}$,

$$I(P) - I_q(P) \leqslant \ln \frac{1}{q} , \tag{3}$$

where $I(P) = -\sum_{n=0}^{\infty} p_n \ln p_n$. However, we did not show whether the upper bound in (3) is achievable and if it is not, whether there is a tighter bound. The

0019-9958/80/040034-03$02.00/0

object of this note is to prove that the upper bound in (3) cannot be improved and is in fact the least upper bound for the difference $I(P) - I_q(P)$.

2. Proof of the Result

Let $m = \sum_{n=0}^{\infty} np_n$ be the mean number of points in X for the original point process. Provided that $m < \infty$, one can deduce that both $I(P)$ and $I_q(P)$ exist and are finite. In the sequel we assume $0 < m < \infty$. The proof of our result hinges upon the following theorem.

Theorem 1. *Amongst all initial distributions $\{p_n\}$ with fixed mean m, the difference $I(P) - I_q(P)$ for fixed $q \in (0, 1)$ is maximal iff*

$$p_n = \left(\frac{m}{m+1}\right)^n \left(\frac{1}{m+1}\right), \qquad n = 0, 1, \ldots . \tag{4}$$

Proof. Let $I_q'(P)$ denote the first derivative of $I_q(P)$ with respect to q. Zeephongsekul (1978) has shown that for each $q \in (0, 1)$,

$$I_q'(P) = q^{-1} \sum_{n=1}^{\infty} p_{q,n} \ln \left(\frac{p_{q,n-1}}{p_{q,n}}\right)^n. \tag{5}$$

For any $q \in (0, 1)$, $p_{q,n} = 0$ implies $p_{q,m} = 0$ for every $m \geqslant n$ by (1). Therefore each term in (5) is well defined since $0 \ln a/0 = 0$ if $a \geqslant 0$.

By the Mean Value Theorem of differential calculus,

$$\begin{aligned} I(P) &= I_{(1-q)+q}(P) \\ &= I_q(P) + (1-q)\, I_\xi'(P) \end{aligned} \tag{6}$$

for some $\xi \in (q, 1)$. Therefore the proof will be completed once we show that the distribution given by (4) maximizes $I_\xi'(P)$. The upper bound for $I_\xi'(P)$ is obtained by applying Jensen's inequality (Feller, 1971, p. 153) to (5) resulting in

$$I_\xi'(P) \leqslant m \ln \left(1 + \frac{1}{m\xi}\right)$$

with equality iff

$$\frac{p_{\xi,n-1}}{p_{\xi,n}} = \left(1 + \frac{1}{m\xi}\right). \tag{7}$$

The recurrence relation (7) now implies

$$p_{\xi,n} = \left(\frac{m\xi}{m\xi + 1}\right)^n \left(\frac{1}{m\xi + 1}\right), \qquad n = 0, 1,\ldots$$

and hence $p_n = \lim_{\xi \uparrow 1} p_{\xi,n} = (m/m + 1)^n(1/m + 1)$ as asserted.

By Theorem 1, the search for the tighest upper bound of $I(P) - I_q(P)$ for fixed $q \in (0, 1)$ over all initial distributions $\{p_n\}$ reduces to that of finding the tightest bound amongst the geometric distributions given by (4). In this case, using Example 1 in Zeephongsekul (1978),

$$\begin{aligned} I(P) - I_q(P) &= (m + 1)\ln(m + 1) - m \ln m - (mq + 1)\ln(mq + 1) + mq \ln mq \\ &= \ln \left(1 + \frac{1}{m}\right)^m \left(1 + \frac{1}{mq}\right)^{-mq} \left(\frac{m + 1}{mq + 1}\right). \end{aligned} \tag{8}$$

Since the function $g(x) = (x + 1) \ln (x + 1) - x \ln x$ is concave and strictly increasing, the least upper bound of (8) occurs when $m \to \infty$ and equals $\ln 1/q$. Hence our result is proved.

RECEIVED: November 2, 1979

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